XI. Spin Angular Momentum or Simply Spin

- Focus on Electron's Spin Angular Momentum
  [but "spin" can be assigned to particles other than electron]
- Why "Spin"?
  - Need it to understand periodic table
    e.g. p orbitals ($\ell = 1$, $m_\ell = 1, 0, -1 \Rightarrow 3$ states)
    but periodic feature among elements suggested
    p orbitals have $3 \times \underbrace{2}_{\text{from spin}} = 6$ states
  - Early experiments asked for it, in addition to orbital AM
  - Need it to explain high-precision hydrogen spectrum

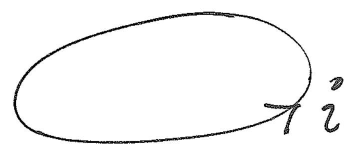
A. Concept on Experimental Measurement of Angular Momentum

- Discussed Orbital AM $\quad L^2 \to l(l+1)\hbar^2, \quad l = 0, 1, 2, \dots$

  $L_z \to m_l\hbar, \quad m_l = \underbrace{l, l-1, \dots, -l}_{(2l+1)\text{ values}}$

- How to do experiments on measuring Orbital AM?
  - Classical picture

$\Rightarrow$

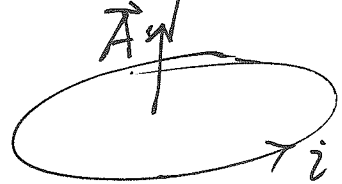


orbiting $\underbrace{\text{electron}}_{\text{charge } -e}$ $\Rightarrow$ current loop

$\Downarrow$

$\vec{\mu}$ = magnetic dipole moment

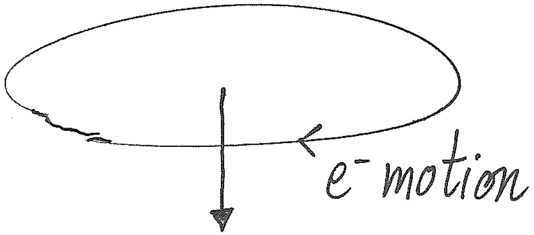
$= i\vec{A}$

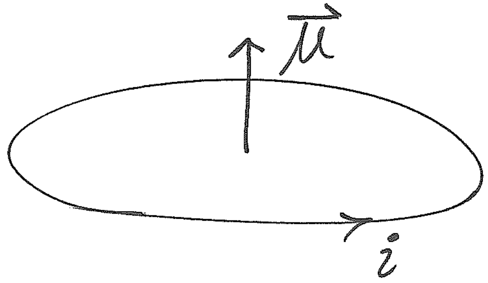


$\vec{A}$ → Magnitude = Area of loop

$\vec{A}$ → direction: $\perp$ plane of loop

Thus,





$\vec{L}$ = Orbital AM

($\because$ electron is negatively charged)

$\therefore$ Expected $\boxed{\vec{\mu}_L \propto -\vec{L}}$ (1)

Magnetic dipole moment due to orbital AM (subscript "L")

Current $\rightarrow$ $|i| = e\,\dfrac{v}{2\pi r}$ [charge passing by a point per unit time]

$$\left.\begin{aligned} |\vec{\mu}_L| &= |i|\cdot \pi r^2 = \frac{1}{2} e v r \\ |L| &= r\, m_e v \end{aligned}\right\} \Rightarrow \frac{|\vec{\mu}_L|}{|L|} = \frac{e}{2m_e} \qquad (2)$$

Angular Momentum $\rightarrow$ $|L| = r\, m_e v$

Putting (1) & (2) together:

for an electron $\rightarrow$ $\boxed{\vec{\mu}_L = \dfrac{-e}{2m_e}\vec{L}}$ (3)

This is "Think Classical"

"Go Quantum"

$$\boxed{\hat{\vec{\mu}}_L = -\underbrace{\frac{e}{2m_e}}_{\text{a constant}} \hat{\vec{L}}} \qquad (4)$$

Operator

opposite directions

Copy result: eigenvalues of $|\vec{\mu}_L|$

$$|\vec{\mu}_L| = \frac{e}{2m_e}|\vec{L}| = \underbrace{\frac{e\hbar}{2m_e}}_{\text{a combination of } \overbrace{e,\hbar,m_e}^{\text{constants}}}\sqrt{\ell(\ell+1)}$$

$$= \mu_B \sqrt{\ell(\ell+1)} \qquad (5)$$

$$\mu_B \equiv \frac{e\hbar}{2m_e} = \text{Bohr Magneton} = \underbrace{\begin{cases} 9.274 \times 10^{-24}\ \text{J/Tesla} \\ 5.79 \times 10^{-5}\ \text{eV/Tesla} \end{cases}}_{\text{[Note order of magnitude]}}$$

$$(\mu_L)_z = z\text{-component} = \frac{-e}{2m_e}L_z \ \text{ takes on } \ \frac{-e}{2m_e} m_\ell\hbar = -\mu_B \cdot m_\ell$$

- But QM deals with Energy (starts with Hamiltonian)
  - Given that $\vec{L} \to \vec{\mu}_L$, how to form an energy term?

$$\boxed{\vec{\mu}_L \text{ and magnetic field } \vec{B} \text{ interact}}$$

$$\boxed{U_{magnetic} = -\vec{\mu}_L \cdot \vec{B}} \quad (6) \quad \text{in an external applied } \vec{B}$$

an additional term in $\hat{H}$

- Recall: Atom (spherically symmetric $U(r)$)

  $\Rightarrow$ No idea about what x, y, z directions are about!

- Applied $\vec{B}$ field $\Rightarrow$ $\vec{B}$ defines $\underbrace{\text{a direction}}_{\text{call it } z\text{-direction}}$

  i.e. $\vec{B} = B\hat{z}$ (no loss in generality) for a state of quantum # m_ℓ

$$\therefore\ U_{magnetic} = -\vec{\mu}_L \cdot B = -(\mu_L)_z B \stackrel{\downarrow}{=} +\frac{e\hbar}{2m_e} B\, m_\ell = \mu_B \cdot B \cdot m_\ell$$

[$B \sim$ few Tesla, $\mu_B \sim 10^{-5}$ eV/Tesla, $U_{magnetic} \sim 10^{-4} - 10^{-5}$ eV tiny but detectable]

<u>Take-Home Message</u>

- electron has charge $(-e) \to \vec{L}$ leads to $\vec{\mu}_L = \frac{-e}{2m_e}\vec{L}$
- $\vec{\mu}_L$ interacts with $\vec{B} \Rightarrow U_{magnetic} = -\vec{\mu}_L \cdot \vec{B} = -(\mu_L)_z B$ (7)
- To do experiments on orbital AM, play with applied $\vec{B}$ field
- $\vec{B} = 0$ (p-orbitals) / $\vec{B} \neq 0$ (p-orbitals)

$\vec{B} = 0$ (p-orbitals)		$\vec{B} \neq 0$ (p-orbitals)	m_ℓ
Y_{11}, Y_{10}, Y_{1-1} (degenerate)	— — — $E_{B=0}$	— $(E_{B=0} + \mu_B B)$	$+1$
		— $E_{B=0}$	0
		— $(E_{B=0} - \mu_B B)$	-1

"Removed the degeneracy"

- This is the physics behind the $\underbrace{\text{Zeeman Effect}}_{}$, (next course)

one spectral line ($\vec{B}=0$) splits into several ($\vec{B}\neq 0$)
(1902 Nobel Prize)

- Note that $\underbrace{m_\ell = -1}_{}$ has lowest energy

$\vec{\mu}_L$ tends to be "aligned" with $\vec{B}$ $\left((\mu_L)_z = -\mu_B m_\ell = +\mu_B \right)$

and thus $\vec{L}$ tends to be "anti-aligned" with $\vec{B}$ $\left(L_z = m_\ell \hbar = -\hbar \right)$

- <u>Important for discussion on Spin</u>

s-orbital (e.g. H-atom in ground state)

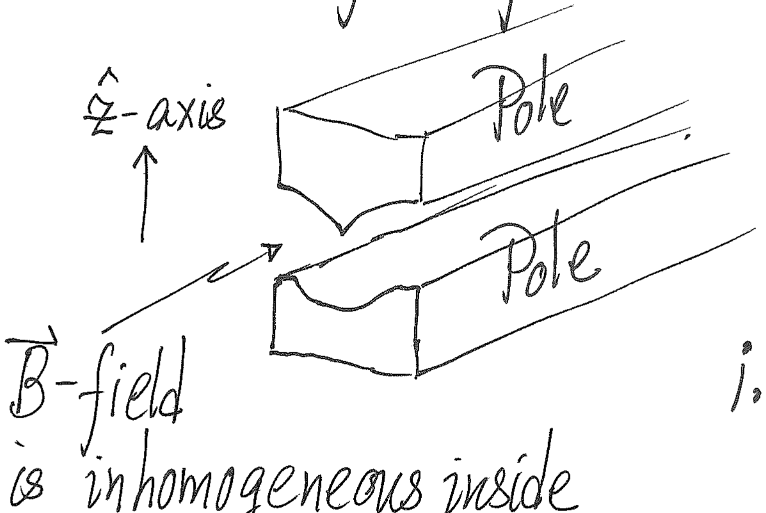
$\Rightarrow \ell = 0 \Rightarrow m_\ell = 0$ only

$\Rightarrow$ Don't expect to observe any effect in $\vec{B}$-field <u>if</u> there is <u>only</u> orbital AM (8)

But this is <u>NOT</u> what experiments showed! (See Sec. C)

B. Inhomogeneous Magnetic Field exerts force on magnetic dipole moment

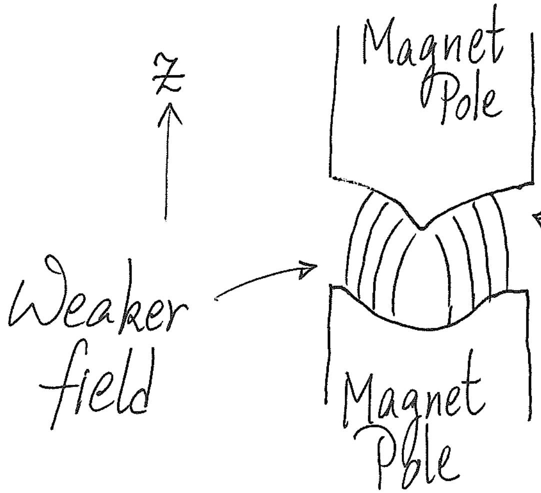
- An example of inhomogeneous (non-uniform) magnetic field
  [Uniform field: $\vec{B}(x,y,z) = \vec{B}$ (same $\vec{B}$ at different places)]



i.e. $\vec{B}(\vec{r})$

Special designed magnet for giving inhomogeneous field

- So what?
  - A magnetic dipole moment $\vec{\mu}$ feels a force in an inhomogeneous $\vec{B}$-field



Inhomogeneous $\vec{B}$-field

[denser field lines $\Rightarrow$ stronger field]

$B(z) \Rightarrow$ Non-uniform
(B is different at different places)

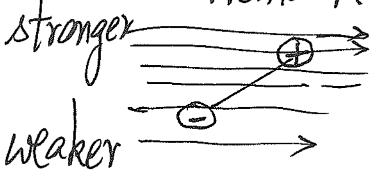
$$\vec{F} = \text{Force}^{+} \text{ on } \vec{\mu} \text{ due to inhomogeneous } \vec{B}\text{-field}$$

$$= -\vec{\nabla}(-\vec{\mu}\cdot\vec{B}) \qquad (-\vec{\nabla}(\text{potential energy}) = \text{force})$$

$$= (\vec{\mu}\cdot\vec{\nabla})\vec{B}$$

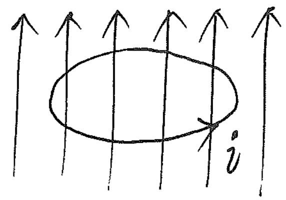
$\therefore$ As long as $\vec{B}$ varies, there is a force on $\vec{\mu}$.

$^{+}$Remark: There is an electric dipole moment in an inhomogeneous $\vec{E}$-field analogy.

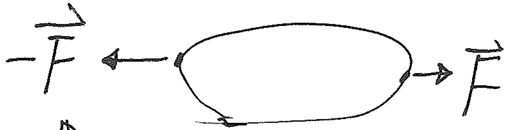


net force on $\vec{p}_{electric} \sim (\vec{p}_{el.}\cdot\vec{\nabla})\vec{E}$

Classical picture: $\vec{\mu} \approx$ current loop



in Uniform field



cancel
(no net force)

[Recall: $\underbrace{(-e)\vec{v}}_{\sim \vec{J}} \times \vec{B}$ = force on moving charge]

$\sim \vec{J} \times \vec{B}$

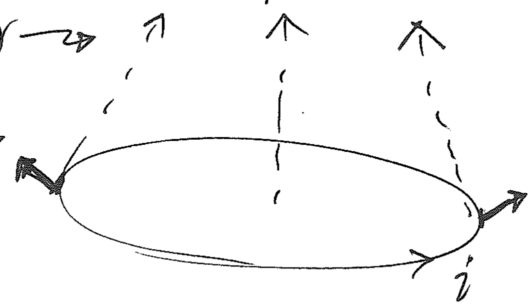
current density

Non-uniform Field

$\frac{\partial B}{\partial z} \neq 0$

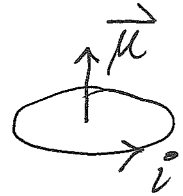
stronger

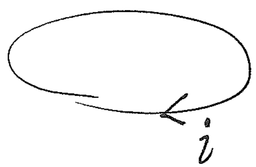
weaker



Forces only cancel in the plane of loop

But Net Force $F_z \neq 0$

$\Rightarrow$ Net Force in direction of $\frac{\partial B}{\partial z}$ for 

For 

Net Force in opposite direction

$F_z \propto \mu_z \frac{\partial B}{\partial z}$ $\Rightarrow$ different μ_z experiences different forces

Example: Hydrogen atoms (all excited to 2p states)

- Only knowledge of orbital AM ($l=1$, $m_l = 1, 0, -1$)

$L_z = \hbar, 0, -\hbar$

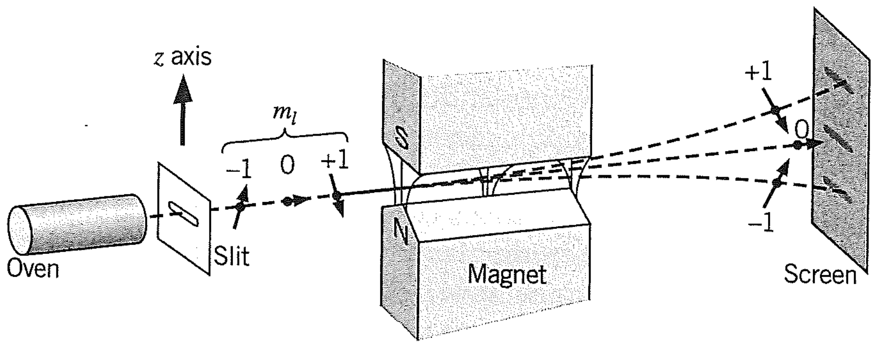
- Passing atoms through inhomogeneous $\vec{B}$-field [assuming all stay in 2p]

$(\mu_L)_z = -\mu_B, 0, +\mu_B$

No force

opposite forces

We would expect to see



Schematic diagram of Stern-Gerlach experiment. A beam of atoms from an oven passes through a slit and then enters a region where there is a nonuniform magnetic field. Atoms with their magnetic dipole moments in opposite directions experience forces in opposite directions.

[Taken from Krane, "Modern Physics"]

Key Point

- Different μ_z: different forces
- This experimental set up is called the Stern-Gerlach Experiment

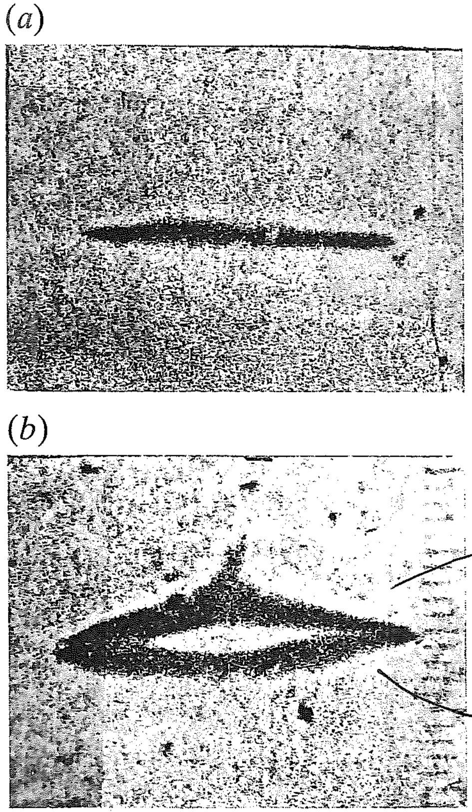
C. The Stern-Gerlach Experiment

- 1922 using Ag atoms (1 outer-electron in s-orbital) ($l=0$)
- 1927 (Phipps and Taylor) using hydrogen atoms (1s) ($l=0$)
- Pass beam of atoms through Stern-Gerlach set up
- Orbital AM ($l=0$), $(\mu_L)_z = 0 \Rightarrow$ No effect from orbital AM
- Thus, did <u>not</u> expect splitted beams even $\vec{B} \neq 0$
- Moreover, any effect due to orbital AM gives odd number of splitted beams (e.g. $l=1$, 3 beams)
- But this was <u>NOT</u> what Stern and Gerlach observed!

Classic results of Stern† & Gerlach (1922)

$\vec{B} = 0$

Turn On $\vec{B}$



The results of the Stern-Gerlach experiment. (*a*) The image of the slit with the field turned off. (*b*) With the field on, two images of the slit appear on the screen. The scale at right represents 1 mm.

One Up
[Two beams]
One down

[Not odd number]

Here comes Spin Angular Momentum

[Recall: $l = 0$]

Interpretation

- $\vec{\mu}$ due to something other than orbital AM
- μ_z can take on only two possible values

$\Rightarrow m_j = 1/2, -1/2$ only

$\therefore j = 1/2$ only (one value)

† Stern was awarded the Nobel Prize in 1943, after 82 nominations since 1925! Gerlach was not awarded, probably due to political reason.

Experimental Facts

- Atoms come out as two split beams $\begin{cases} \text{some atoms feel an upward force} \\ \text{some atoms feel a downward force} \end{cases}$

- But orbital AM plays no role ($\because$ $l=0$ (s-orbital))

- Yet, atoms seem to carry a magnetic dipole moment with two eigenvalues for μ_z (z-component) ($\because$ 2 beams)

- This $\vec{\mu}$ does not come from $\vec{L}$

- With the idea that $\vec{\mu}$ is associated with an angular momentum (and it stems from the electron), there must be another AM that comes from the electron — the spin angular momentum of electron.

- Just like general AM, the same results (two beams come out) show up in any "direction" you place the Stern-Gerlach set up. [No special direction]
- Any direction, always see two beams (two $m_j = 1/2, -1/2$ values)
- Consistent with: Can know J^2 and one component (called J_z)